Finite Element Solution of Open Region Electrostatic Problems Incorporating the Measured Equation of Invariance

Griffin K. Gothard, Sadasiva M. Rao, *Senior Member, IEEE*, Tapan K. Sarkar, *Fellow, IEEE*, and Magdalena Salazar Palma, *Member, IEEE*

***Abstract*—In this work, we utilize the finite element technique to open region problems in conjunction with the truncation condition based on the measured equation of invariance (MEI) concept. The major advantage of the present scheme is a significant reduction in the number of unknowns while retaining the sparsity of the generating matrix. Typical numerical results are presented for the solution of Laplace's equation to illustrate the accuracy of the technique.**

I. Introduction

In THIS paper, the finite element (FE) technique [1] in conjunction with the measured equation of invariance (MEI) [2] has been used to analyze Laplace's equation for two-dimensional electrostatic problems involving open regions, i.e.

$$\nabla^2 \mathbf{V} = \frac{\partial^2 \mathbf{V}(x,y)}{\partial x^2} + \frac{\partial^2 \mathbf{V}(x,y)}{\partial y^2} = 0 \qquad x,\, y \in [0,\, \infty]. \tag{1}$$

The principal advantage of the "MEI" boundary condition is that it allows the truncation of the grid space very close to the body while retaining the accuracy/advantages of conventional difference methods.

We may mention here that the application of MEI to electrostatics has been previously discussed [3]. However, in this paper we provide a more efficient approach. The essential differences are as follows:

1) The FE grid is truncated two layers from the body in all cases.
2) Only three neighboring nodes are used for computing weighting coefficients using MEI method.
3) For two-body problem, a single "umbilical chord" is used.

The organization of the paper is as follows: In the next section, we describe our grid termination technique. In Section III, some typical numerical examples are provided for comparison purposes. Since the FE method is well-known, the description of the method is not provided in this paper.

Manuscript received January 27, 1995. This work was supported by the NASA Langley Research Center.

G. Gothard and S. Rao are with the Department of Electrical Engineering, Auburn University, Auburn, AL 36849 USA.

T. Sarkar is with the Department of Electrical Engineering, Syracuse University, Syracuse, NY 13244-1240 USA.

M. Palma is with ETS de Ingenieros de Telecomunicacion, Ciudad Universitaria, 28040 Madrid, Spain.

IEEE Log Number 9412818.

II. Termination of FE Grid Using MEI Method

The MEI method presented in [2] utilizes a six point difference equation relating the nodes on the boundary to the interior nodes, by

$$\sum_{j=1}^{M} \beta_j \mathbf{V}_j = 0 \quad (\text{Here } \mathbf{M} = 6). \tag{2}$$

For a boundary node, the adjacent five points were chosen as shown in Fig. 1. It has been assumed that a difference equation like [2] actually simulates an essentially numerical absorbing boundary condition. In this case, we set $\beta_1 = 1$ without any loss of generality [2]. The other values of β_j are found by assuming five different linearly independent fixed charge distributions [e.g., 1, $\cos(x)$, $\sin(x)$, $\cos(2x)$, $\sin(2x)$] on the structure. Based on these charge distributions, the potential at all the boundary nodes and at any interior layer adjacent to the boundary layer may be computed utilizing the free space Green's function. Thus, for two-dimensional static regions, we get

$$\mathbf{V}(\rho) = \int_c \frac{q_s(\rho') \ln |\rho - \rho'|}{2\pi\epsilon_o}\, dc' \tag{3}$$

where c is the perimeter of the object, and $q_s(\rho')$ is the assumed charge density on the contour. So once the five potentials at the same points ($\mathbf{V}_2 - \mathbf{V}_6$ of Fig. 1) are known, $\beta = 2$ to 6 can be solved for by solving a 5×5 matrix equation. These values of β_j are sufficient for relating the boundary node to its neighbors. Note that in this scheme, the sparsity of the matrix is still maintained since only six points are used [2].

In this work, M has been chosen as 4. Since a finite difference solution with MEI uses only four nodes (three boundary nodes and one interior node), it is felt that same number is sufficient for the FE/MEI solution also. Therefore for node $\mathbf{V}_1$, of Fig. 1, we choose $\mathbf{V}_2$, $\mathbf{V}_4$, and $\mathbf{V}_6$ to enforce the absorbing boundary condition. This is accomplished by choosing three independent charge distributions given by $q_s^1(l) = 1.0$, $q_s^2(l) = \sin(2\pi l/\mathbf{L})$, and $q_s^3(1) = \cos(2\pi l/\mathbf{L})$ where $\mathbf{L}$ is the total circumferencial length of the object, l is the parameter measured along the contour of the two-dimensional body and

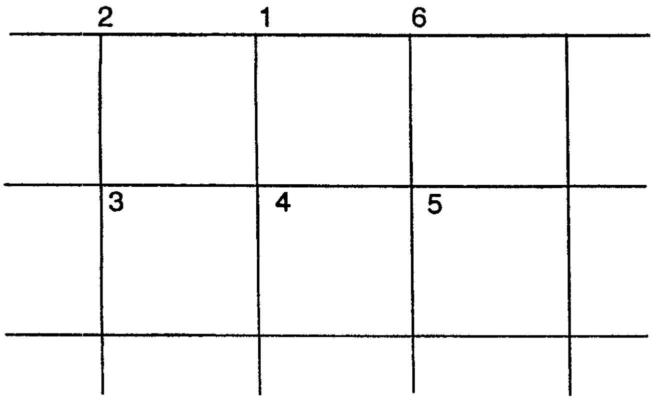

Fig. 1. Boundary node number 1 and nearest neighbors.

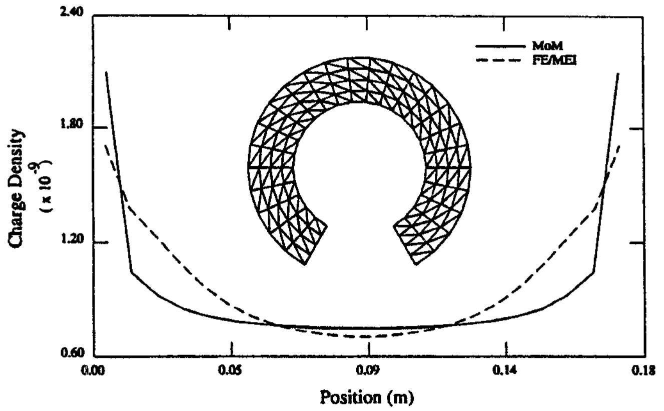

Fig. 2. The charge distribution on an infinite PEC bent circular strip raised to a 10-V potential as predicted by FE/MEI and MoM. The bent strip has a radius of 0.4 m and an arc length of 0.56π.

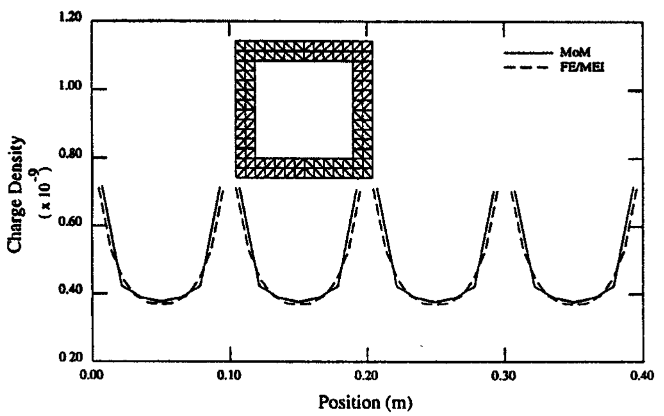

Fig. 3. The charge distribution on an infinite PEC square cylinder 0.1 m to a side and raised to 10 V potential as predicted by FE/MEI and MoM.

$q_s(l)$ is the surface charge density. By substituting these charge distributions in (3), one can determine numerical values for the potentials at each of the node points at the two outermost layers. Substituting these values in (2) with M = 4 one solves for the three different unknowns (β_j's) associated with each of the outermost nodes.

Further, in this work, we observed that only two layers are sufficient to obtain accurate results. Although it is possible

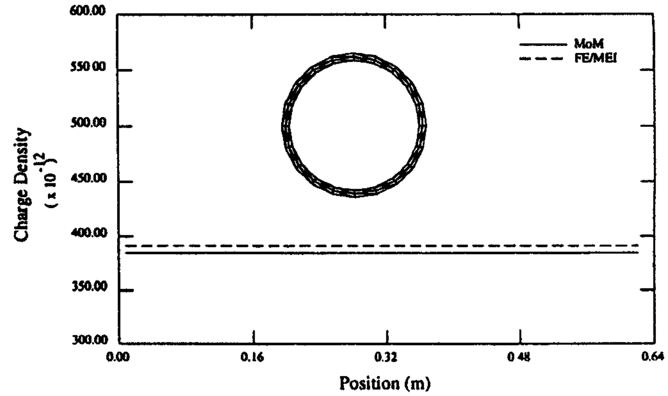

Fig. 4. The charge distribution on an infinite PEC circular cylinder 0.1 m to a side and raised to 10-V potential as predicted by FE/MEI and MoM. The radius of the cylinder is 0.1 m.

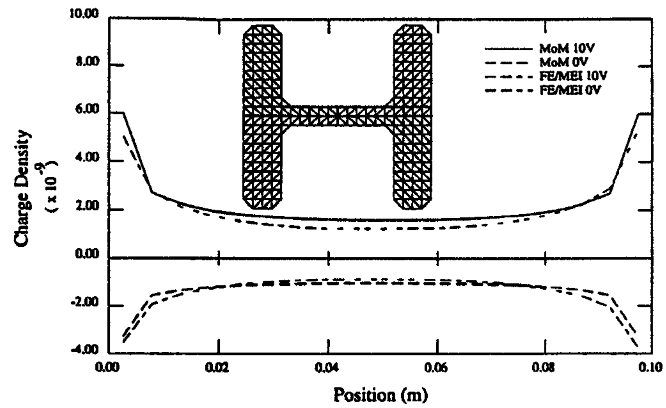

Fig. 5. The charge distribution on two infinite PEC strips 0.1 m wide and separated by 0.1 m as predicted by FE/MEI and MoM. One strip is raised to a 10-V potential while the other is held to a 0-V potential.

to obtain more accurate results by increasing the number of layers, we do not recommend this procedure since it results in a large number of unknowns and thus inefficient. In fact, our suggestion is to increase the grid density while keeping two layers only, which will increase the number of unknowns only marginally while providing greater object resolution. We think this is an important point to note while using the FE/MEI method.

III. Numerical Results

In this section, the charge distribution for a bent circular strip, a rectangular cylinder, a circular cylinder, and two straight strips are presented in Figs. 2–5, respectively. For all cases presented, the results are compared with the method of moments (MoM) [4] solution for accuracy purposes. Further, the grid scheme used for the FE/MEI and MoM solution is presented in the inset of each figure. It may be noted that in each case the FE/MEI results compare favorably with the MoM solution. The small amount of discrepancy in each figure may be attributed to the fact that for the FE/MEI solution the computed charge distribution is a half-cell away from the actual conducting surface.

IV. CONCLUSION

The FE/MEI is presented for solution of the Laplace's Equation in two dimensions for open region problems. It is evident that, for these examples presented, only two layers beyond the body boundary is sufficient to obtain reasonably accurate solution. Further, the "umbilical" technique may be used to link separated bodies thus providing an efficient FE/MEI solution for the multiple body problem.

REFERENCES

[1] J. Jin, *The Finite Element Method in Electromagnetics* New York: Wiley-Interscience, 1993.

[2] K. K. Mei, R. Pous, Z. Chen, Y. W. Liu, and M. D. Prouty, "Measured equation of invariance: A new concept in field computation," *IEEE Trans. Antennas Propagat.*, vol. 42, pp. 320–328, Mar. 1994.

[3] M. D. Prouty, K. K. Mei, S. E. Schwartz, and R. Pous, "A new approach to quasistatic analysis with application to microstrip," *IEEE Microwave and Guided Wave Lett.*, vol. 3, pp. 302–304, Sept. 1993.

[4] R. F. Harrington, *Field Computation by Moment Methods.* New York: Macmillan, 1968.