# Efficient Solution of the Differential Form of Maxwell's Equations in Rectangular Regions

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***Abstract*— One of the problems of the finite element and the finite difference method is that as the dimension of the problem increases, the condition number of the system matrix increases as $\Theta(1/h^2)$ (of the order of $h^2$, where $h$ is the subsection length). Through the use of a suitable basis function tailored for rectangular regions, it is shown that the growth of the condition number can be checked while still retaining the sparsity of the system matrix. This is achieved through a proper choice of entire domain basis functions. Numerical examples have been presented for efficient solution of waveguide problems with rectangular regions utilizing this approach.**

## I. Introduction

The finite difference [1] and the finite element method [2] have been developed over the last few years in the microwave area for efficient solution of the differential form of Maxwell's equations. Researchers have primarily focused their attention on development of basis functions for treating boundaries with edges and open region problems, spurious-free solutions of eigenvalue problems, and efficient solution of sparse matrix equations.

However, one of the problems with the finite difference and finite element method lies in the solution of a large matrix equation (either a direct solution with several right-hand sides, or an eigenvalue problem). The problem here is that as the number of basis functions increases (and hence the dimension and size of the matrix), the condition number of the matrix also increases. The increase of the condition number of the matrix creates various types of solution problems. For example, the condition number directly dictates the solution procedure as a highly ill-conditioned matrix prohibits application of a direct matrix solver like Gaussian Elimination [3], [4], and more sophisticated techniques like singular value decomposition may have to be introduced [5]. There are various ways to stop the increase of the condition number as the dimension of the matrix increases. One such procedure has been outlined by Mikhlin [4]. In [4], the basis functions are chosen in such a way that the growth of the condition number can be controlled. In this paper, we utilize a particular set of basis functions primarily tailored for rectangular regions for an efficient solution of the resulting matrix equation. This particular choice of the basis is related to the "wavelet" concepts [6]–[8].

The basic philosophy of this paper then lies in the choice of a particular set of basis functions (which of course is dependent on the nature of the problem, e.g., TM or TE and on particular shape of the domain) which attempts to diagonalize the system matrix that arises when Galerkin's method is applied to the differential form of Maxwell's equations. The ideal situation will of course be to make the large sparse "Galerkin System Matrix" diagonal. Then the solution of such a matrix (either solution of the matrix equation due to different right hand sides or solution of an eigenvalue problem) problem would be trivial. However, because of various boundary conditions, this goal cannot be achieved. Therefore, the next best procedure is an attempt to make say 80% of the sparse system matrix $[S]$ diagonal. So for a 21 × 21 system matrix $[S]$, we would have an 18 × 18 matrix block that is diagonal, so the solution of a 21 × 21 matrix equation is simply to reduce to inversion to a 3 × 3 matrix $[B]$ and two 3 × 18 and 18 × 3 matrices as illustrated below:

$$[S]_{21\times 21} \equiv \begin{bmatrix} [D]_{18\times 18} & [G]^*_{18\times 3} \\ [G]_{3\times 18} & [B]_{3\times 3} \end{bmatrix}$$

where * denotes conjugate transpose.

For obtaining the cutoff frequencies of a waveguide, one needs to solve for the eigenvalues of a large matrix equation. However, if the matrix is sparse and "almost" diagonal, then an iterative technique like the conjugate gradient [9] can be utilized to converge on the first few of the eigenvalues in a relatively few iterations to yield the cutoff frequencies. Some numerical examples are presented to illustrate the problem.

The main contribution of this paper is that for rectangular regions, a suitable basis can be found which produces a system matrix which has a large diagonal block. The size of the block increases as the dimension of the problem increases. Hence, the system matrix can then be subdivided into a square matrix of relatively small dimension which is sparse, and two sparse rectangular matrices and a large diagonal matrix. This improves the computational efficiency of the new technique over conventional finite element methods as the condition number of the system matrix does not increase significantly as the dimension of the problem increases.

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## II. Solution of Helmholtz's Equation Utilizing the New Basis

Consider the solution of Helmholtz equation

$$\nabla^2 u(x, y) + k^2 u(x, y) = F(x, y). \quad (1)$$

We focus our attention on the particular case, when the domain of (1) is restricted to rectangular regions $L$ defined by the contour $\Gamma$. In this paper, we have focused our attention only on rectangular regions, and have assumed that any arbitrary shaped region $R$ can be made of rectangular regions $L$ of the type

$$L: \quad 0 \le x \le a \quad \text{and} \quad 0 \le y \le b. \quad (2)$$

To solve (1) in the region (2), one multiples (1) by the function $v(x)$ and integrates over the region $L$ to obtain

$$\int_L v(x, y)\,\nabla^2 u(x, y)\,dx\,dy + k^2 \int_L u(x, y) v(x, y)\,dx\,dy = \int_L F(x, y) v(x, y)\,dx\,dy. \quad (3)$$

After integrating by parts, the result is

$$-\int_L (\nabla v)(\nabla u)\,dx\,dy + k^2 \int_L uv\,dx\,dy = \int_L Fv\,dx\,dy + \int_L v\frac{du}{dn}\,d\Gamma \quad (4)$$

where $n$ is the direction of the outward normal.

Next, it is assumed that the unknown $u(x, y)$ can be represented by a complete set of basis functions, which have first-order differentiability, as

$$u(x, y) \simeq \hat{u} = \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij}\phi_{ij}(x, y) + \sum_{i=1}^{4}\sum_{J=1}^{P} B_{ij}N_{ij}(x, y) + \sum_{i=1}^{4} C_i T_i(x, y) \quad (5)$$

where $A_{ij}$, $B_{ij}$, and $C_{ij}$ are the unknowns to be solved for.

Basically, the function $\phi_{ij}(x, y)$ satisfy the homogeneous boundary conditions, and $N_{ij}$ and $T_i$ are there to take care of the inhomogeneous Dirichlet conditions and enforcing continuity of the fields from one rectangular region to the next.

Specifically, for the waveguide problems involving rectangular regions, the basis functions have been chosen in the following form:

$$\phi_{ij}(x, y) = \sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right). \quad (6)$$

The rationale for choosing these specific basis functions is the fact that these functions are not only orthogonal to themselves, but their partial derivatives are also orthogonal in the rectangular region defined in (2), i.e.,

$$\langle \phi_{ij}; \phi_{pq}\rangle = 0 \quad \text{for} \quad i \ne p; j \ne q \quad (7)$$

and

$$\langle \nabla\phi_{ij}; \nabla\phi_{pq}\rangle = 0 \quad \text{for} \quad i \ne p; j \ne q \quad (8)$$

$$\langle \phi_{ij}; \nabla\phi_{pq}\rangle = 0 \quad \text{for} \quad i \ne p; j \ne q \quad (9)$$

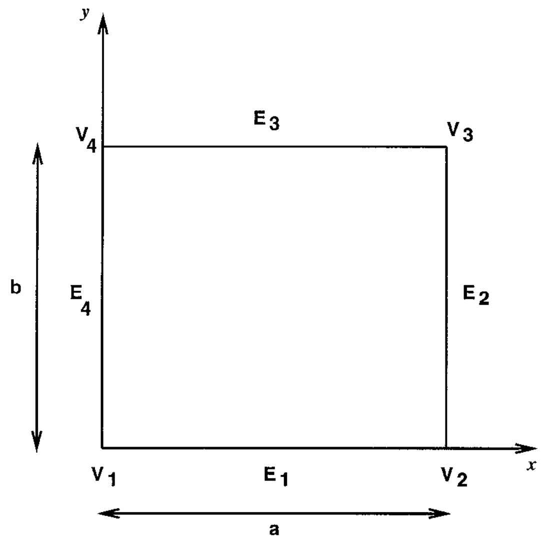


Fig. 1. Geometry for the 2-D basis functions.

where the inner product in the two-dimensional rectangular region $R$ is defined by the usual Hilbert inner product

$$\langle c; d\rangle = \int_0^a dx \int_0^b dy\, c(x, y)\,\overline{d}(x, y) \quad (10)$$

where the overbar denotes complex conjugate.

In addition, we need four edge basis functions $N_{ij}$, where $N_{ij}$ is zero everywhere on the boundary (i.e., on all edges) except on edge $E_i$. This implies $N_{3j}$ is zero on edges $E_1$, $E_2$, and $E_4$ and unity on edge $E_3$ (as in Fig. 1). Hence,

$$N_{1j}(x, y) = \left(1 - \frac{y}{b}\right)\sin\left(\frac{j\pi x}{a}\right) \quad (11)$$

$$N_{2j}(x, y) = \frac{x}{a}\sin\left(\frac{j\pi y}{b}\right) \quad (12)$$

$$N_{3j}(x, y) = \frac{y}{b}\sin\left(\frac{j\pi x}{a}\right) \quad (13)$$

$$N_{4j}(x, y) = \left(1 - \frac{x}{a}\right)\sin\left(\frac{j\pi y}{b}\right). \quad (14)$$

In an analogous fashion, one can illustrate that the basis $T_i$ in (5) provide the matching conditions needed for the four vertices $V_1$, $V_2$, $V_3$, and $V_4$ as shown in Fig. 1. Specifically, the basis associated with each vertex can be written as

$$T_1(x, y) = \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right) \quad (15)$$

$$T_2(x, y) = \frac{x}{a}\left(1 - \frac{y}{b}\right) \quad (16)$$

$$T_3(x, y) = \frac{x}{a}\cdot\frac{y}{b} \quad (17)$$

$$T_4(x, y) = \left(1 - \frac{x}{a}\right)\frac{y}{b}. \quad (18)$$

Substitution of (6), (11)–(14), (15)–(19) in (5) and then into (4), with $v(x, y)$ replaced by $\phi_{ij}$, $N_{ij}$, and $T_i$ results in a matrix equation of the form

$$[P][A] + k^2[Q][A] = [V_F] + [V_B]. \quad (19)$$

TABLE I
CUTOFF WAVENUMBERS OF THE $TM_{mn}$ MODES

| | $TM_{11}$ | $TM_{21}$ | $TM_{31}$ | $TM_{12}$ | $TM_{41}$ & $TM_{22}$ | $TM_{32}$ | $TM_{51}$ | $TM_{42}$ |
|---|---|---|---|---|---|---|---|---|
| N=1 | 3.52 | 4.45 | * | * | * | * | * | * |
| N=2 | 3.51 | 4.45 | 5.66 | 6.48 | * | * | * | * |
| N=3 | 3.51 | 4.44 | 5.66 | 6.48 | 7.05 | 7.85 | * | * |
| N=4 | 3.51 | 4.44 | 5.66 | 6.48 | 7.04 | 7.85 | 8.51 | 8.90 |
| N=5 | 3.51 | 4.44 | 5.66 | 6.48 | 7.03 | 7.85 | 8.48 | 8.89 |
| N=6 | 3.51 | 4.44 | 5.66 | 6.48 | 7.03 | 7.85 | 8.47 | 8.89 |
| N=7 | 3.51 | 4.44 | 5.66 | 6.48 | 7.03 | 7.85 | 8.47 | 8.89 |
| N=8 | 3.51 | 4.44 | 5.66 | 6.48 | 7.03 | 7.85 | 8.46 | 8.89 |
| N=9 | 3.51 | 4.44 | 5.66 | 6.48 | 7.03 | 7.85 | 8.46 | 8.89 |
| N=10 | 3.51 | 4.44 | 5.66 | 6.48 | 7.03 | 7.85 | 8.46 | 8.89 |
| EXACT | 3.51 | 4.44 | 5.66 | 6.48 | 7.02 | 7.85 | 8.46 | 8.89 |

Because of a special choice of the basis functions tailored for rectangular domains, the system matrices $[P]$ and $[Q]$ have a certain structure, namely,

$$[P];\,[Q] = \begin{bmatrix} [D]_{n\times n} & [G]^{*}_{n\times m} \\ [G]_{m\times n} & [B]_{m\times m} \end{bmatrix} \tag{20}$$

where $[D]$ is a diagonal matrix, $[B]$ is a sparse matrix, $[G]$ is a sparse matrix, and $[G]^*$ is its conjugate transpose. $[V_F]$ is a vector containing the excitation terms. $[V_B]$ is a vector containing the boundary terms. The percentage of the matrix that is diagonal depends, first, on how many rectangular regions the original region has been divided into and, second, the nature of the boundary condition on the contour $\Gamma$.

If the original domain $R$ has been subdivided into $L$ secondary rectangular regions, then the continuity of the function $u$ is imposed along all boundary edges, and at each vertex through the coefficients $B_{ij}$ and $C_i$. In addition, the continuity of the first derivative of $u$ in the normal direction to the $L$ subdomain boundaries is also enforced. This condition is imposed by making the boundary terms in the formulation from one subdomain to its connecting neighbor equal.

Depending on the type of the boundary condition—either Dirichlet or Neumann—the structure of the system matrices $[P]$ and $[Q]$ are different. We now consider the structure of the system matrices as a function of the boundary condition.

*Case A—Dirichlet:* For this case, where the original domain has been subdivided into $L$ regions and the highest order of approximation $M, N$, and $P$ in (5) has been assumed to be the same, all $N$, i.e., they are considered to be the same in all $L$ regions for comparison purposes.

Because of the special choice of the basis in (5), the maximum dimension of the system matrix $[P]$ and $[Q]$ will be $L(N^2+4N+4)$. However, if the boundary conditions are strictly homogeneous, then the total dimension of the system matrices $[P]$ and $[Q]$ will be somewhat less than $L(N^2 + 4N + 4)$. However, for the choice of the special basis, the dimension of the diagonal submatrix $[D]$ in (20) will be $LN^2$. This clearly demonstrates that as the number of unknowns $N$ increase, the majority of the system matrix becomes diagonal.

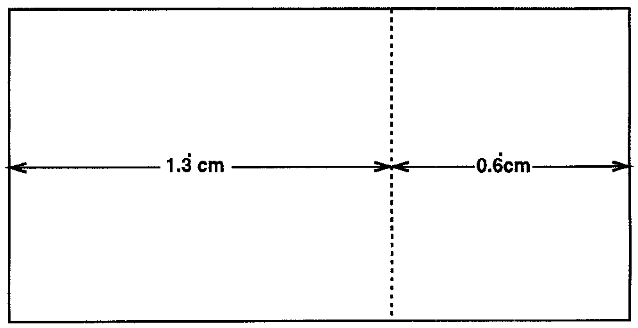


Fig. 2. Rectangular waveguide.

This is because the row size increase of $[P]$ is dominated by the term $LN^2$, and so is the row size of the diagonal matrix $[D]$. The rectangular submatrix $[G]$ has the dimension of rows as ($N$ times the number of internal edges + number of internal corners) and the column is $LN^2$. The square matrix $[B]$ has a row and column dimension of ($N$ times the number of internal edges + number of internal corners). Hence, the size of $B$ goes up as essentially $\theta(L + 1)N$. Therefore, the computational complexity goes up as $\theta[\{(L + 1)N\}^3]$, when the number of unknowns go up by $LN^2$. This amounts to a significant decrease in the reduction of computational complexity.

*Case B—Neumann:* For this case, the diagonal submatrix is the same size as that for the previous case of Dirichlet boundary conditions. But now, the coefficients of all the matching functions are unknowns. Hence, the size of the system matrix $[P]$ is $LN^2 + N^*$ {(number of edges) + (number of corners)}. Even though the size of the diagonal matrix is the same as before, case B produces a system matrix $P$ which is almost diagonal.

*Case C—Mixed:* It is easy to extrapolate the results to a mixed Dirichlet and Neumann condition. The important point is that due to the choice of the "tailored" basis, the major portion of the system matrix $[P]$ and $[Q]$ is diagonal.

Because a large portion of the system matrix is diagonal, the growth of the condition number with the increase in the number of unknowns can be controlled by proper scaling.

TABLE II
PERCENTAGE OF THE MATRIX THAT IS DIAGONAL FOR THE TM MODES

| | Matrix size | Size of the Diagonal Block | % Diagonal |
|---|---|---|---|
| N=1 | 3 | 2 | 66.7% |
| N=2 | 10 | 8 | 80% |
| N=3 | 21 | 18 | 85.7% |
| N=4 | 36 | 32 | 88.9% |
| N=5 | 55 | 50 | 90.9% |
| N=6 | 78 | 72 | 92.3% |
| N=7 | 105 | 98 | 93.3% |
| N=8 | 136 | 128 | 94.1% |
| N=9 | 171 | 162 | 94.7% |
| N=10 | 210 | 200 | 95.2% |

TABLE III
CUTOFF WAVENUMBERS OF THE $TE_{mn}$ MODES

| | $TE_{10}$ | $TE_{20}$ & $TE_{01}$ | $TE_{11}$ | $TE_{21}$ | $TE_{30}$ | $TE_{31}$ | $TE_{40}$ & $TE_{02}$ | $TE_{12}$ |
|---|---|---|---|---|---|---|---|---|
| N=1 | 1.59 | * | * | * | * | * | * | * |
| N=2 | 1.57 | 3.16 | 3.53 | 4.46 | * | * | * | * |
| N=3 | 1.57 | 3.15 | 3.53 | 4.46 | 4.75 | 5.70 | * | * |
| N=4 | 1.57 | 3.14 | 3.52 | 4.45 | 4.74 | 5.69 | 6.32 | 6.53 |
| N=5 | 1.57 | 3.14 | 3.51 | 4.44 | 4.72 | 5.67 | 6.30 | 6.49 |
| N=6 | 1.57 | 3.14 | 3.51 | 4.44 | 4.72 | 5.67 | 6.30 | 6.49 |
| N=7 | 1.57 | 3.14 | 3.51 | 4.44 | 4.72 | 5.67 | 6.29 | 6.48 |
| N=8 | 1.57 | 3.14 | 3.51 | 4.44 | 4.72 | 5.67 | 6.29 | 6.48 |
| N=9 | 1.57 | 3.14 | 3.51 | 4.44 | 4.71 | 5.66 | 6.29 | 6.48 |
| N=10 | 1.57 | 3.14 | 3.51 | 4.44 | 4.71 | 5.66 | 6.29 | 6.48 |
| EXACT | 1.57 | 3.14 | 3.51 | 4.44 | 4.71 | 5.66 | 6.28 | 6.48 |

TABLE IV
PERCENTAGE OF THE MATRIX THAT IS DIAGONAL FOR THE TE MODES

| | Matrix size | Size of the Diagonal Block | % Diagonal |
|---|---|---|---|
| N=1 | 15 | 2 | 13.3% |
| N=2 | 28 | 8 | 28.6% |
| N=3 | 45 | 18 | 40% |
| N=4 | 66 | 32 | 48.5% |
| N=5 | 91 | 50 | 54.9% |
| N=6 | 120 | 72 | 60% |
| N=7 | 153 | 98 | 64.0% |
| N=8 | 190 | 128 | 67.4% |
| N=9 | 231 | 162 | 70.1% |
| N=10 | 276 | 200 | 72.5% |

## III. APPLICATION TO SOME WAVEGUIDE PROBLEMS

As a first example, consider the solution of the cutoff frequencies of the various TE and TM modes of a rectangular waveguides of dimension 2 cm × 1 cm. Here we could have used one region ($L = 1$) to solve the problem and increase $N$. But to illustrate the flexibility and accuracy of the procedure, we divide the rectangular region into two regions $A^{(1)}$ $1.\dot{3} \times 1$ and $A^{(2)}$ $0.\dot{6} \times 1$ as shown in Fig. 2. The basis chosen is the same as in Section II.

TABLE V
CUTOFF WAVENUMBERS FOR THE TM MODES OF AN L-SHAPED WAVEGUIDE

| | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 | Mode 7 |
|---|---|---|---|---|---|---|---|
| N=1 | 4.95 | 6.18 | * | * | * | * | * |
| N=2 | 4.92 | 6.15 | 7.00 | 8.63 | * | * | * |
| N=3 | 4.91 | 6.14 | 7.00 | 8.58 | 8.94 | 10.2 | 10.6 |
| N=4 | 4.90 | 6.14 | 7.00 | 8.57 | 8.92 | 10.2 | 10.6 |
| N=5 | 4.90 | 6.14 | 7.00 | 8.56 | 8.91 | 10.1 | 10.6 |
| N=6 | 4.90 | 6.14 | 7.00 | 8.56 | 8.91 | 10.1 | 10.6 |
| N=7 | 4.89 | 6.14 | 7.00 | 8.56 | 8.90 | 10.1 | 10.6 |
| N=8 | 4.89 | 6.14 | 7.00 | 8.56 | 8.90 | 10.1 | 10.6 |
| Ref [9] | 4.80 | 6.07 | 6.92 | 8.61 | | | |
| Ref [12] | 4.87 | 6.13 | 6.99 | 8.55 | | | |

TABLE VI
PERCENTAGE OF THE MATRIX THAT IS DIAGONAL FOR THE TM CASE

| | Dimension of System Matrix | Dimension of the Diagonal Block | % Diagonal |
|---|---|---|---|
| N=1 | 5 | 32 | 60% |
| N=2 | 16 | 12 | 75% |
| N=3 | 33 | 27 | 81.8% |
| N=4 | 56 | 48 | 85.7% |
| N=5 | 85 | 75 | 88.2% |
| N=6 | 120 | 108 | 90% |
| N=7 | 161 | 147 | 91.3% |
| N=8 | 208 | 192 | 92.3% |

Table I presents the cutoff wavenumbers of the $TM_{mn}$ mode. The * indicates that the order was not sufficient to perform reliable computation for the modes for a rectangular waveguide with better than 1% accuracy. The exact solution is obtained from [11]. Table II indicates the percent of the matrix that is diagonal. The computational efficiency of the new basis now becomes clear. For this case, 95% of the matrix is diagonal when a large number of unknowns are taken, and as the number of unknowns increase so does the size of the diagonal matrix! Table III presents the cutoff wavenumbers for the $TE_{mn}$ modes in a rectangular waveguide. The exact solution is obtained from [11]. Table IV shows that as the number of unknowns increases, so does the size of the diagonal block maintaining the computational efficiency. For the same value of $N$, the size of the matrix is different for the TE case as opposed to the TM case because many of the boundary terms go to zero for the TM case and not for the TE case.

As a second example, we consider an L-shaped waveguide as shown in Fig. 3. The largest dimensions are all 1 cm. The structure has been subdivided into three subregions. This problem has been solved using a finite difference approximation [9] and an integral equation approach [12]. The results produced by this new approach are accurate and convergence is very rapid.

TABLE VII
CUTOFF WAVENUMBERS FOR THE TE MODES OF AN L-SHAPED WAVEGUIDE

| | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 |
|---|---|---|---|---|---|---|
| N=1 | 1.89 | 2.92 | * | * | * | * |
| N=2 | 1.88 | 2.90 | 4.85 | 5.21 | 5.49 | 6.87 |
| N=3 | 1.88 | 2.89 | 4.85 | 5.21 | 5.48 | 6.87 |
| N=4 | 1.87 | 2.89 | 4.84 | 5.20 | 5.47 | 6.84 |
| N=5 | 1.87 | 2.89 | 4.84 | 5.20 | 5.46 | 6.84 |
| N=6 | 1.87 | 2.89 | 4.83 | 5.20 | 5.46 | 6.84 |
| N=7 | 1.87 | 2.89 | 4.83 | 5.20 | 5.46 | 6.84 |
| N=8 | 1.87 | 2.89 | 4.83 | 5.20 | 5.46 | 6.84 |
| N=9 | 1.87 | 2.89 | 4.83 | 5.20 | 5.46 | 6.48 |
| Ref [9] | 1.88 | 2.95 | 4.89 | 5.26 | 5.49 | 6.91 |
| Ref [12] | 1.89 | 2.91 | 4.87 | 5.24 | | |

Table V provides the cutoff wavenumbers for first few dominant TM modes for the L-shaped waveguide. Table VI shows that, as usual, as the number of unknowns increase, so does the size of the diagonal block. Table VII provides the cutoff wavenumbers for the first few TE modes of the L-shaped waveguide. Again, the percentage of the matrix that is

TABLE VIII
PERCENTAGE OF THE MATRIX THAT IS DIAGONALIZABLE FOR THE TE CASE

| | Dimension of System Matrix | Dimension of the Diagonal Block | % Diagonal |
|---|---|---|---|
| N=1 | 21 | 3 | 14.3% |
| N=2 | 40 | 12 | 30% |
| N=3 | 65 | 27 | 41.5% |
| N=4 | 96 | 48 | 50% |
| N=5 | 133 | 75 | 56.4% |
| N=6 | 176 | 108 | 61.4% |
| N=7 | 225 | 147 | 65.3% |
| N=8 | 280 | 192 | 68.6% |
| N=9 | 341 | 243 | 71.3% |

TABLE IX
CUTOFF WAVENUMBERS OF THE TE MODES OF A VANED RECTANGULAR WAVEGUIDE

| | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 |
|---|---|---|---|---|---|---|
| N=1 | 1.60 | 2.21 | 3.32 | 3.55 | 4.58 | * |
| N=2 | 1.57 | 2.17 | 3.17 | 3.31 | 4.26 | 5.41 |
| N=3 | 1.57 | 2.15 | 3.15 | 3.30 | 4.26 | 4.75 |
| N=4 | 1.57 | 2.14 | 3.15 | 3.30 | 4.25 | 4.74 |
| N=5 | 1.57 | 2.13 | 3.14 | 3.30 | 4.25 | 4.72 |
| N=6 | 1.57 | 2.12 | 3.14 | 3.30 | 4.25 | 4.72 |
| N=7 | 1.57 | 2.12 | 3.14 | 3.30 | 4.25 | 4.72 |
| N=8 | 1.57 | 2.12 | 3.14 | 3.30 | 4.25 | 4.72 |
| N=9 | 1.57 | 2.12 | 3.14 | 3.30 | 4.25 | 4.71 |
| N=10 | 1.57 | 2.21 | 3.14 | 3.30 | 4.25 | 4.71 |
| Ref. [9] | 1.57 | 2.00 | 3.13 | 3.28 | 4.23 | 4.66 |
| Ref [12] | 1.57 | 2.11 | 3.16 | 3.30 | | |

TABLE X
PERCENTAGE OF THE MATRIX THAT IS DIAGONALIZABLE FOR THE TE CASE

| | Dimension of System Matrix | Dimension of the Diagonal Block | % Diagonal |
|---|---|---|---|
| N=1 | 27 | 4 | 14.8% |
| N=2 | 52 | 16 | 30.8% |
| N=3 | 85 | 36 | 42.3% |
| N=4 | 126 | 64 | 50.8% |
| N=5 | 175 | 100 | 57.1% |
| N=6 | 232 | 144 | 62.1% |
| N=7 | 297 | 196 | 66% |
| N=8 | 370 | 256 | 69.2% |
| N=9 | 451 | 324 | 71.8% |
| N=10 | 540 | 400 | 74.0% |

diagonal increases consistently with the number of unknowns as shown in Table VIII.

As a final example, consider the vaned rectangular waveguide shown in Fig. 4. Table IX provides the cutoff wavenumbers of the first few dominant TE modes. The results have been compared to that of finite difference solution technique [9] and an integral equation technique [12]. Again, as the number of unknowns increase, the majority of the system matrix is diagonal as shown in Table X, and hence the computational efficiency increases.

TABLE XI
CUTOFF WAVENUMBERS OF THE TM MODES OF A VANED RECTANGULAR WAVEGUIDE

| | Mode 1 | Mode 2 | Mode 3 | Mode 4 | Mode 5 | Mode 6 |
|---|---|---|---|---|---|---|
| N=1 | 3.74 | * | * | * | * | * |
| N=2 | 3.72 | 5.05 | 6.48 | * | * | * |
| N=3 | 3.71 | 5.03 | 6.48 | 6.59 | 7.03 | 7.87 |
| N=4 | 3.71 | 5.01 | 6.48 | 6.55 | 7.03 | 7.80 |
| N=5 | 3.71 | 5.01 | 6.48 | 6.53 | 7.03 | 7.78 |
| N=6 | 3.70 | 5.00 | 6.48 | 6.52 | 7.02 | 7.76 |
| N=7 | 3.70 | 5.00 | 6.48 | 6.51 | 7.02 | 7.76 |
| N=8 | 3.70 | 4.99 | 6.48 | 6.50 | 7.02 | 7.76 |
| N=9 | 3.70 | 4.99 | 6.48 | 6.50 | 7.02 | 7.75 |
| N=10 | 3.70 | 4.99 | 6.48 | 6.50 | 7.02 | 7.75 |
| Ref. [9] | 3.65 | 4.87 | 6.31 | | | |

TABLE XII
PERCENTAGE OF THE MATRIX THAT IS DIAGONALIZABLE FOR THE TM CASE

| | Dimension of System Matrix | Dimension of the Diagonal Block | % Diagonal |
|---|---|---|---|
| N=1 | 27 | 10 | 37.0% |
| N=2 | 52 | 27 | 51.9% |
| N=3 | 85 | 52 | 61.2% |
| N=4 | 126 | 85 | 67.5% |
| N=5 | 175 | 126 | 72.0% |
| N=6 | 232 | 175 | 75.4% |
| N=7 | 297 | 232 | 78.1% |
| N=8 | 370 | 297 | 80.3% |
| N=9 | 451 | 370 | 82.0% |
| N=10 | 540 | 451 | 83.5% |

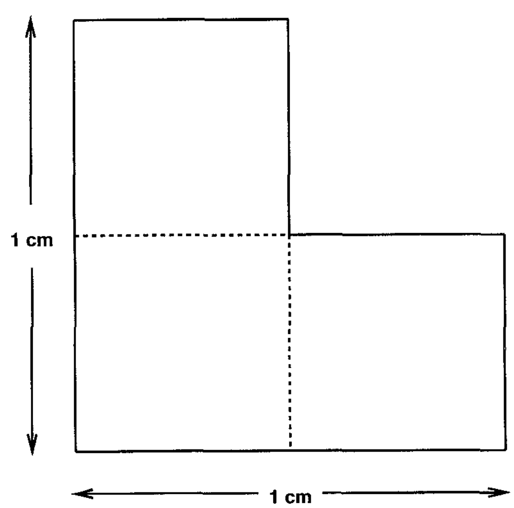

Fig. 3. L-shaped waveguide.

Finally, Table XI provides the cutoff wavenumbers of TM modes of a vaned rectangular waveguide, and Table XII shows the size of the system matrix that is diagonal.

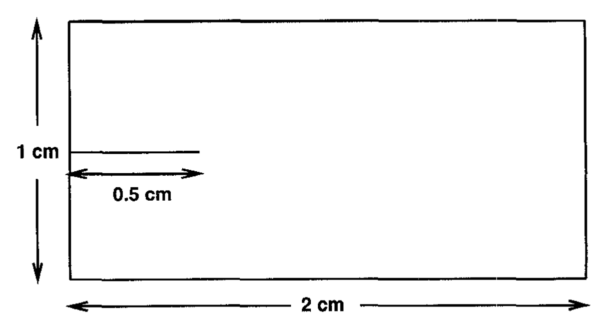

Fig. 4. Vaned rectangular waveguide.

## IV. CONCLUSION

An entire domain basis function is presented for efficient solution of a Helmholtz equation confined to two-dimensional rectangular regions. Since this particular choice of the basis function transforms the majority of the system matrix into a diagonal one, the growth of condition number can easily be controlled by proper scaling, and the computational efficiency can be significantly enhanced over the conventional technique.

This principle has been applied to the cutoff wavenumbers of certain waveguides. The rate of convergence and accuracy of the new basis is reasonable. In addition, these basis functions can easily be extended to 3-D rectangular regions and to problems with dielectric inhomogeneity.

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